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Equations For Isentropic Compressible Flows In Variable Area Ducts

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ABSTRACT
Analysis of compressible flow in variable area ducts can be carried out using simple algebraic equations: no need to read M from tables. Availability of modern computers makes compressible flow analysis quite simple and straight forward.

Keywords
Compressible flow; governing equations; variable area ducts.

1. INTRODUCTION
Analysis of compressible flows in ducts is invariably done by introducing the parameter M (Mach number). For solution of real life problems one needs to refer to various tables of precalculated values. There is a fundamental problem with this approach: one may determine Mach number at a location but will need additional information like temperature to determine the velocity. Also interpolation is usually necessary to arrive at exact figures which can potentially introduce errors in calculations.

Consider, for instance, the example 5.1 of ref.1. Reservoir pressure and temperature are given and the air flows through a convergent-divergent nozzle. At a location in the convergent section where nozzle area is 6 times the throat area, first of all the value of M is read from a table. Then two ratios (one for pressures and the other for the temperatures) are read from another table. Then the actual pressure and actual temperature are determined by multiplying the ratios with stagnation values. Similar procedure is followed for the divergent section by reading values from different tables.

It is possible to analyze the problem in a simple way starting from initial conditions and proceeding downstream using standard laws. This is the subject matter of this communication.

2. BASIC EQUATIONS
Equation of state: $p = \rho RT$ where ρ is density of the fluid, p is the absolute pressure and T is the absolute temperature. R is the gas constant (287 J/kg.K for air).

Ratio of specific heats $\gamma = c_p/c_v$ where $c_p = \gamma R/(\gamma - 1)$

and $c_v = R/(\gamma - 1)$. For air, $c_p = 1004$ J/kg.K and $\gamma = 1.4$.

Continuity equation: $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$ or

$\rho_1/\rho_2) = (A_2/A_1)(V_2/V_1)$

(V_1 and V_2 are velocities, m/s)

From equation of state:

$$T_2 = (p_2/p_1)(A_2/A_1)(V_2/V_1).T_1 \quad (1)$$

Steady state energy equation (neglecting potential energy):

$$\frac{V_2^2}{2} + \frac{\gamma}{\gamma-1}RT_2 = \frac{V_1^2}{2} + \frac{\gamma}{\gamma-1}RT_1$$

Or $$V_2^2 + Z.T_2 = V_1^2 + Z.T_1 \quad (2)$$

where $Z = 2\gamma R/(\gamma-1) = 2009$ for air.

For isentropic flow: $\frac{p_1}{\rho_1^\gamma} = \frac{p_2}{\rho_2^\gamma}$

or $$\left(\frac{p_1}{p_2}\right) = \left(A_2/A_1\right)^\gamma \left(V_2/V_1\right)^\gamma \quad (3)$$

From equations 1, 2 and 3,

$$\left(\frac{V_2}{V_1}\right)^2 + Z.\left(\frac{V_2}{V_1}\right)^{(1-\gamma)}\left(\frac{A_1}{A_2}\right)^{(\gamma-1)}\frac{T_1}{V_1^2} = 1 + \frac{Z.T_1}{V_1^2} \quad (4)$$

Similarly following equations can be derived

$$\left(\frac{A_1}{A_2}\right)^2\left(\frac{T_1}{T_2}\right)^{\frac{2}{(\gamma-1)}} + Z.\frac{T_2}{V_1^2} = 1 + \frac{Z.T_1}{V_1^2} \quad (5)$$

$$\left(\frac{A_1}{A_2}\right)^2\left(\frac{p_1}{p_2}\right)^{\left(\frac{2}{\gamma}\right)} + \frac{Z.T_1}{V_1^2}\left(\frac{p_2}{p_1}\right)^{\frac{(\gamma-1)}{\gamma}} = 1 + \frac{Z.T_1}{V_1^2} \quad (6)$$

$$\left(\frac{A_1}{A_2}\right)^2\left(\frac{\rho_1}{\rho_2}\right)^2 + \frac{Z.T_1}{V_1^2}\left(\frac{\rho_2}{\rho_1}\right)^{(\gamma-1)} = 1 + \frac{Z.T_1}{V_1^2} \quad (7)$$

Equations 4 to 7 are for isentropic compressible flow in a variable area duct. Velocity at any location in the duct can be calculated from eqn 4, temperature from eqn 5, pressure from eqn 6 and density from eqn 7. Help of spreadsheet like Excel is needed to determine the values ('goal seek' needs to be used). Velocity of sound can be calculated from the knowledge of γ, R and T and Mach number can be calculated as the ratio of fluid velocity and velocity of sound.

3. EXAMPLES
Consider a converging nozzle with a diameter of 100 mm at the inlet and 70 mm somewhere downstream. Two cases are analyzed: first with velocity of 100 m/s at the inlet and, second, with inlet velocity of 10 m/s (see chart 1 cells B4 and D4). The value of right hand side of equations 4 to 7 is 63.8817 (cell B12) for the first case and 6289.17 (cell D12) for the second case.

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	A	B	C	D	E
1	gamma	1.4		1.4	
2	R	287		287	
3					
4	Velocity, V1 m/s	100		10	
5	Temperature, T1, C	40		40	
6	Temperature, T1 K	313		313	
7	Area, A1 sq m	0.007854		0.00785398	
8	Pressure, P1 Pascal	101000		101000	
9	Density, Ro1 kg/cub	1.1243335		1.12433347	
10					
11	factor	2009		2009	
12	RHS	63.8817		6289.17	
13					
14	c1, m/s	354.63136		354.631358	
15	dia,mm	100	70	100	70
16	Area, A2 sq m	0.007854	0.0038	0.00785398	0.003848
17	A1/A2	1	2.0408	1	2.040816
18	Velocity, V2	99.998494	258.74	9.99988047	20.4337
19	LHS	63.882049	63.882	6289.20004	6289.2
20	Incompressible v	100	204.08	10	20.40816
21	ratio comp/incom v	0.9999849	1.2678	0.99998805	1.001252
22	Temperature, T2	313.00001	284.66	313.000009	312.8434
23	LHS	63.881702	63.882	6289.17017	6289.2
24	Temperature T2, C	40.000009	11.656	40.0000086	39.84344
25	Pressure, P2	101000.02	72452	101001.687	100823.3
26	LHS	63.881704	63.882	6289.19999	6289.2
27					
28	Density, Ro2	1.1243334	0.8869	1.12434693	1.122928
29	LHS	63.881698	63.882	6289.20009	6289.2
30					
31	c2	354.63136	338.19	354.631363	354.5427
32	M	0.2819787	0.7651	0.02819796	0.057634

Chart 1:results of calculations for low and high velocities.

Velocity V2 is determined from eqn 4 in such a way that the value of the LHS of eqn 4 matches with the value of RHS; this can be done by using 'goal seek' function of spreadsheet. It can be seen that when the inlet velocity is 10 m/s, the velocity at the second section is 20.43 m/s (cell E18). When the velocity at first section is 100 m/s, the velocity at second section would be 258.7 m/s (cell C18).

Temperature at second section would be 312.8 K (cell E22) (hardly different from that at first section) for the 10 m/s case whereas it would be 284.7K (cell C22) for the 100 m/s case (substantially different from the figure of 313K at first section).

Pressure at second section would be 100823 Pa (cell E25) for the 10 m/s case whereas it would be 72452 Pa (cell C25) for the 100 m/s case.

Density at second section would be 1.123 kg/m^3 (cell E28) (hardly different from 1.124 kg/m^3 at first section) for the 10 m/s case whereas it would be 0.88 kg/m^3 (cell C28) for the 100 m/s case.

Mach number for the 10 m/s case varies from 0.0282 (cell D32) to 0.058 (cell E32). For the 100 m/s case, it varies from 0.282 (cell B32) to 0.765 (cell C32).

All the above calculations have been done without resorting to reading value of Mach number as a first step.

Consider another example: Let example 5.1 of ref.1 be modified as follows. Consider the isentropic subsonic-supersonic flow through a convergent-divergent nozzle. There are two locations in the nozzle where A/A* = 6; one in the convergent section and the other in the divergent section. Let pressure and temperature in the reservoir be 10 atm and 300 K (velocity being assumed zero).

Let subscript 0 denote the point in the reservoir, 1 denote the location of the section in convergent section, 2 denote the throat and 3 denote the section in the divergent section. Areas at sections 1 and 3 are six times the area at section 2. Velocity at section 1 is given as 33.6 m/s. Calculate velocity, temperature, pressure, density and M at each location.

Information at location 0 is fully known. P_0 =10 atm (1032500 Pa); T_0=300 K. From this information we can calculate density $\rho_0 = P_0/R.T_0$ where R is the gas constant (287 kJ/kg.K for air). This gives ρ_0 = 11.992 kg/m^3.

Since velocity is known at section 1, temperature at section 1 can be calculated.

$T_1 = T_0 – V_1^2/(2.C_p)$ which gives T_1= 299.44K.

From two equations ($\rho_1 = p_1/RT_1$ and $p1 = p_0 - \rho_1.V_1^2/2$), p_1 and ρ_1 can be calculated.

P_1 = 1025762 Pa and ρ_1 = 11.93607 kg/m^3.
Thus c1 = 346.8619 m/s; M1 = 0.096869.

Table 1: Comparison of values as per reference 1 and this communication.

Location	Item	Value as per ref.1	Value as per this communication
1	Velocity, m/s	33.6 (given)	33.6 (given)
	Temperature, K	299.4	299.4355
	Pressure, atm	9.94	9.9347
	Density, kg/m^3	§	11.9361
	Speed of sound, m/s	§	346.8619
	Mach No	0.097	0.096869
2	Velocity, m/s	§	306.525
	Temperature, K	§	253.23
	Pressure, atm	§	5.5262
	Density, kg/m^3	§	7.8503
	Speed of sound, m/s	§	318.98
	Mach No	§	0.960955
3	Velocity, m/s	646.7	646.84
	Temperature, K	91.77	91.73
	Pressure, atm	0.154	0.1581
	Density, kg/m^3	§	0.62
	Speed of sound, m/s	192	191.98
	Mach No	3.368	3.369
§ - Not given			

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Information at location 2 can be calculated from equations 4 to 7 specifying $A_2/A_1 = 1/6$.

$V_2 = 306.525$ m/s; $T_2 = 253.23$ K; $P_2 = 570575.8$ Pa (5.5262 atm); $\rho_2 = 7.8503$ kg/m^3; c2 = 318.98 m/s; M2 = 0.960955.

Again information at location 3 can be obtained from equations 4 to 7 by taking A_3/A_1 as 1.

V3 = 646.84 m/s; T3 = 91.732K; P3 = 16323.1 Pa (0.1581 atm); c3 = 191.98 m/s; M3 = 3.369.

Thus the information in summary is given in table 1.

4. AREA RATIO LIMITS

An examination of equations 4 to 7 shows that V_1 and T_1 are the only parameters having absolute values; rest everything (A_2/A_1, V_2/V_1, T_2/T_1, P_2/P_1 and ρ_2/ρ_1) is in ratio form. All equations have two solutions: one for the subsonic region and the other for the supersonic region. Solution of the equations is real only for area ratio A_2/A_1 more than a critical value. Consider, for instance, eqn.4 from which we can calculate V_2/V_1 for various values of area ratio A_2/A_1. Taking $V_1 = 10$ m/s and $T_1 = 313$ K we get curve 2 in fig.1 which shows the plot of V_2/V_1 vs. A_2/A_1. That the equation 4 has two solutions for each value of A_2/A_1 is also apparent from the fig.1.

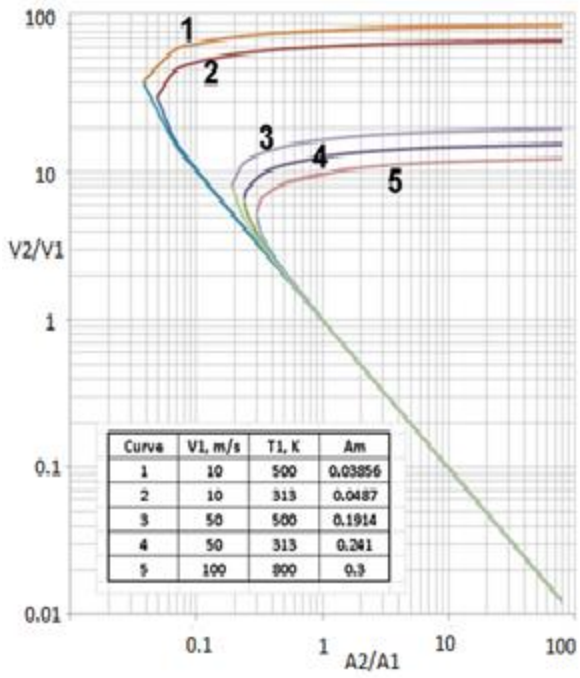

Fig.1: Variation of V2/V1 w.r.t A2/A1 for various values of V1 and T1

Minimum value of ratio A_2/A_1 for which we get real solution for curve 2 is 0.0487; let this be denoted by A_m. The value of A_m depends only upon values of V_1 and T_1.

The value of A_m can be approximated by the empirical equation $A_m = 0.0862V_1/\sqrt{T_1}$. Somehow, A_m seems to represent choked flow. Before making any calculations, one should check the value of A_m and make sure that the value of ratio A_2/A_1 is not smaller than A_m.

5. TEMPERATURE, PRESSURE AND DENSITY

Equations 5, 6 and 7 also have two solutions for each value of the ratio A_2/A_1 above A_m. Fig.2 shows variation of temperature, pressure and density ratios (T_2/T_1, P_2/P_1 and Ro_2/Ro_1) as functions of A_2/A_1 for subsonic region. Similarly fig.3 shows variation of these ratios as a function of A_2/A_1 for the supersonic region.

6. INCOMPRESSIBLE VS COMPRESSIBLE FLOW

Fluid flows are usually classified as compressible and incompressible flows depending upon velocities involved.

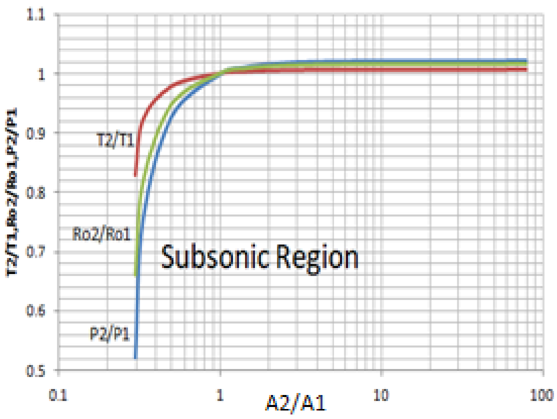

Fig.2: Variation of temperature, pressure and density ratios (T2/T1, P2/P1, Ro2/Ro1) for subsonic region.

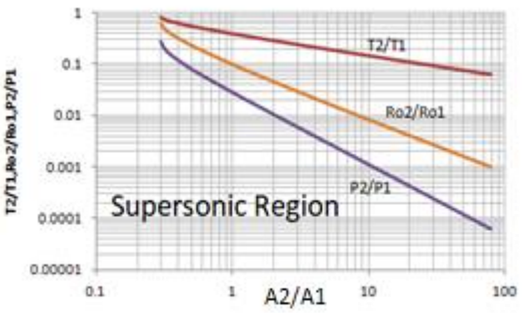

Fig.3: Variation of temperature, pressure and density ratios (T2/T1, P2/P1 and Ro2/Ro1) for supersonic region

While compressible flow is the real life phenomenon, certain assumptions are made for ease of calculations and the flow is called incompressible. It is to be noted that analyses using incompressible flow equations always contain an error, howsoever small. In fact, the incompressible flow has been called a myth[1]. While incompressible analysis may be acceptable in a few practical situations, in some situations it may be desirable to use compressible flow equations to arrive at more accurate results. With availability of simple equations for compressible flows as derived above and computing power of modern day computers, it is no longer necessary to make any assumptions; one can easily solve all duct flow problems as compressible flows.

Venturi meters are often used to measure fluid flow rates and, usually, fluids are considered incompressible. If gas flow rates are being measured by venturi meters, one needs to be careful while assuming fluid to be incompressible particularly for high velocities. It may be better and more accurate if the analysis is carried out using compressible flow equations.

7. CONCLUSION

The usual method of analyzing duct flow problems for compressible fluids need no longer be solved using Mach number tables. Straight forward equations are presented which can be solved using modern computers. The need to interpolate values is dispensed with. Accuracy of results is higher than traditional method.

8. REFERENCE

[1] Modern compressible flow with historical perspective (third edition), John D. Anderson, McGraw Hill.